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Author(s)	Oshima, Yoichi
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On the reachability to some moving domains of time dependent diffusion processes

Yoichi Oshima (Kumamoto University)

1. Preliminaries

Let $X = R^d$, $dm(x) = dx$ and $(\mathcal{E}^t, \mathcal{F})$ be the time dependent Dirichlet forms on $L^2(X; m)$ determined by

$$\mathcal{E}^t(u, v) = \sum_{i,j=1}^d \int_{R^d} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx.$$

Assume that

$$\begin{aligned} \underline{a}(t) \sum_{i,j=1}^d a_{ij}^{(0)}(x) \xi_i \xi_j &\leq \sum_{i,j=1}^d a_{ij}(t, x) \xi_i \xi_j \\ &\leq \frac{1}{\underline{a}(t)} \sum_{i,j=1}^d a_{ij}^{(0)}(x) \xi_i \xi_j \end{aligned}$$

for a non-increasing positive function $\underline{a}(t)$ and positive definite family $(a_{ij}^{(0)}(x))$. $\mathbf{M} = (P_{(s,x)}, X_t)$ be the corresponding diffusion process and $Z_t = (\tau(t), X_t)$ with $\tau(t) = \tau(0) + t$. Fix a domain $\Gamma = \{(t, x) : |x| \geq r(t)\}$ for non-decreasing function $r(t)$ and put $\sigma_\Gamma = \inf\{t > 0 : Z_t \in \Gamma\}$.

Problem

- (1) To give a condition on $r(t)$ under which $P_{(s,x)}(\sigma_\Gamma < \infty) = 1$.
- (2) For a neighbourhood C of 0, to give a condition on $r(t)$ under which

$$\lim_{s \rightarrow \infty} P_{(s, \varphi_0 \cdot m)}(\sigma_\Gamma < \sigma_C) = 0.$$

Note that (2) does not hold if X_t is "transient".

2. General criterion for (1) and its application

Let \widehat{X}_t be the dual process of X_t

$$\begin{aligned}\widehat{Z}_t &= (\widehat{\tau}(t), \widehat{X}_t), \quad \widehat{\tau}(t) = \widehat{\tau}(0) - t \\ \widehat{\sigma}_\Gamma &= \inf\{t > 0 : \widehat{Z}_t \in \Gamma\}, \quad \Gamma \subset [0, \infty) \times X\end{aligned}$$

Then

$$\widehat{u}_\Gamma(t, y) = \widehat{E}_{(t, y)}(\varphi_0(\widehat{X}_t) : t < \widehat{\sigma}_\Gamma)$$

satisfies

$$\int_X \frac{\partial \widehat{u}_\Gamma}{\partial t}(t, x) \varphi(x) dm(x) + \mathcal{E}^t(\varphi, \widehat{u}_\Gamma(t, \cdot)) = 0$$

Let $\Phi(t, \cdot)$ be a one to one smooth mapping from R^d onto itself such that $\Phi(0, y) = y$ and put

$$\begin{aligned}\gamma_{ij}(t, y) &= \frac{\partial \Phi_i}{\partial y_j} \quad (\gamma_{ij}^{-1}) = (\gamma_{ij})^{-1} \\ \alpha_{ij}(t, y) &= \sum_{k,l=1}^d \gamma_{ki}^{-1} a_{kl}(t, \Phi(t)) \gamma_{lj}^{-1} \\ \rho(t, y) &= \det(\gamma_{ij}) \\ \beta_i(t, y) &= \sum_{k=1}^d \gamma_{ki}^{-1} \frac{\partial \Phi_k}{\partial t} \\ \mathcal{E}^{Y,t}(\varphi, \psi) &= \sum_{i,j=1}^d \int_D \alpha_{ij}(t, y) \frac{\partial \varphi}{\partial y_i} \frac{\partial \psi}{\partial y_j} \rho(t, y) dy\end{aligned}$$

Let $Y_t = \Phi(t, X_t)$ $\widehat{Y}_t = \Phi(t, \widehat{X}_t)$. Then they are in dual in $L^2(\mu_t)$, $\mu_t(dy) = \rho(t, y)dy$. For a domain D , put

$$\widehat{\varphi}_F(t, y) = \widehat{P}_{(t, y)}(\varphi_0(\widehat{Y}_t) : t < \sigma_F), \quad F = X \setminus D.$$

Then it satisfies $\widehat{\varphi}_F(0, y) = f(y)$, $\widehat{\varphi}_F(t, y) = 0$ for $y \in F$ and

$$\begin{aligned}& \int_D \psi(t, y) \frac{\partial(\widehat{\varphi}_F \rho)}{\partial t}(t, y) dy + \mathcal{E}^{Y,t}(\psi, \widehat{\varphi}_F) \\ & + \sum_{i=1}^d \int_D \beta_i(t, y) \frac{\partial \psi}{\partial y_i} \widehat{\varphi}_F(t, y) \rho(t, y) dy = 0.\end{aligned}$$

Put $\psi = \widehat{\varphi}_F$ and

$$\widehat{H}_F(t) = \int_{D_0} \widehat{\varphi}_F^2(t, y) \rho(t, y) dy.$$

Then

$$\begin{aligned} \mathcal{E}^{Y,t}(\widehat{\varphi}_F, \widehat{\varphi}_F) &= -\frac{1}{2} \frac{d}{dt} \int_D \widehat{\varphi}^2(t, y) \rho(t, y) dy - \frac{1}{2} \int_D \frac{\partial \rho}{\partial t} \widehat{\varphi}_F^2 dy \\ &\quad - \frac{1}{2} \sum_{i=1}^d \int_D \beta_i(t, y) \frac{\partial \widehat{\varphi}_F^2}{\partial t} \rho(t, y) dy \\ &= -\frac{1}{2} \frac{d\widehat{H}_F}{dt} + \frac{1}{2} \int_D \left(\sum_{i=1}^d \frac{\partial(\beta_i \rho)}{\partial y_i} - \frac{\partial \rho}{\partial t} \right) \widehat{\varphi}_F^2(t, y) dy \\ &\leq -\frac{1}{2} \frac{d\widehat{H}_F}{dt} + \frac{1}{2} \eta(t) \widehat{H}_F(t) \end{aligned}$$

for

$$\eta(t) = \sup_{y \in D} \frac{1}{\rho(t, y)} \left(\sum_{i=1}^d \frac{\partial(\beta_i^{(1)} \rho)}{\partial y_i} - \frac{\partial \rho}{\partial t} \right).$$

Assume that there exists non-increasing function $\bar{\alpha}(t)$ and a positive definite $(\alpha_{ij}^{(0)})$ such that

$$\underline{\alpha}(t) \sum_{i,j=1}^d \alpha_{ij}^{(0)}(y) \xi_i \xi_j \leq \alpha_{ij}(t, y) \xi_i \xi_j.$$

Furthermore assume that there exists a constant $k_0 > 0$ such that

$$k_0 \int_D \varphi(y)^2 dy \leq \sum_{i,j=1}^d \int_D \alpha_{ij}^{(0)}(y) \frac{\partial \varphi}{\partial y_i} \frac{\partial \varphi}{\partial y_j} dy, \quad \forall \varphi \in H_0^1(D) \quad (1)$$

Then

$$\frac{d\widehat{H}_F}{dt} \leq (\eta(t) - 2\lambda(t)) \widehat{H}_F(t)$$

for

$$\lambda(t) = k_0 \underline{\alpha}(t) \frac{\inf_{y \in D} \rho(t, y)}{\sup_{y \in D} \rho(t, y)}.$$

Lemma 1 Under our settings,

$$\widehat{H}_F(t) \leq \|\varphi_0\|_{\mu_0}^2 \exp \left(\int_0^t (\eta(\tau) - 2\lambda(\tau)) d\tau \right). \quad (2)$$

Note that if

$$b(t) \sum_{i,j=1}^d \alpha_{ij}^{(0)}(y) \xi_i \xi_j \leq \sum_{i,j=1}^d a_{ij}^{(0)}(\Phi(t, y)) \xi_i \xi_j$$

for a non-increasing function $b(t)$, then we can take

$$\underline{\alpha}(t) = \frac{1}{r(t)^2} \underline{a}(t) b(t).$$

Put $\Gamma = \{(t, \Phi(t, y)) : t \geq 0, y \in F\}$ and

$$\hat{u}_\Gamma(t, x) = \hat{E}_{(t,x)}^X \left(\varphi_0(\hat{X}_t) : t < \sigma_\Gamma \right).$$

Since $P_{(0, \varphi_0 \cdot m)}^Y(T < \sigma_F) = P_{(0, \varphi_0 \cdot \mu_0)}^Y(T < \sigma_\Gamma)$,

$$\begin{aligned} P_{(0, \varphi_0 \cdot \mu_0)}^Y(T < \sigma_F) &= \int_D \hat{\varphi}_F(t, y) d\mu_T(y) \leq \sqrt{\mu_T(D)} \hat{H}(T)^{1/2} \\ &\leq \|\varphi_0\|_{\mu_0} \sqrt{\mu_T(D)} \exp \left(\int_0^t \left(\frac{1}{2} \eta(\tau) - \lambda(\tau) \right) d\tau \right) \\ &\rightarrow 0, \quad T \rightarrow \infty. \end{aligned}$$

Hence we have

Theorem 2 Under (1), if

$$\lim_{T \rightarrow \infty} \sqrt{\mu_T(D)} \exp \left(\int_0^T \left(\frac{1}{2} \eta(\tau) - \lambda(\tau) \right) d\tau \right) = 0,$$

then $P_{\varphi_0 \cdot m}(\sigma_\Gamma < \infty) = 1$. Apply this theorem for $\Phi_i(t, y) = r(t)y_i$ and $F = \{y : |y| \geq 1\}$. Then

$$\begin{aligned} \gamma_{ij}(t, y) &= r(t) \delta_{ij} \\ \rho(t, y) &= r(t)^d \\ \alpha_{ij}(t, y) &= \frac{1}{r(t)^2} a_{ij}(t, \Phi(t, y)) \\ \lambda(t) &= k_0 \frac{\underline{a}(t)}{r(t)^2} \\ \eta(t) &= \operatorname{div} \beta - d \frac{r'(t)}{r(t)} = 0. \end{aligned}$$

Hence, applying Theorem 2, we have the following theorem.

Theorem 3 Suppose that

$$\lim_{T \rightarrow \infty} r(T)^{1/2} \exp \left(-k_0 \int_0^T \frac{\alpha(\tau)}{r(\tau)^2} d\tau \right) = 0. \quad (3)$$

Then

$$P_{(0, \varphi_0 \cdot m)}(\sigma_\Gamma < \infty) = 1. \quad (4)$$

In particular, if $a_{ij} = \frac{1}{2}\delta_{ij}$, then (3) is satisfied if

$$r(t) \leq c\sqrt{t} \quad (5)$$

for large t with $c < \sqrt{d/(4k_0)}$.

In the case of Brownian motion,

$$\begin{aligned} P_0(|X_t| > r(t)) &= \frac{1}{(\sqrt{2\pi t})^d} \int_{\{|x| > r(t)\}} e^{-|x|^2/(2t)} dx \\ &= \frac{S(1)}{(\sqrt{2\pi})^d} \int_{r(t)/\sqrt{t}}^\infty y^{d-1} e^{-y^2/2} dy \\ &\leq \left(k_1 + k_2 \left(\frac{r(t)}{\sqrt{t}} \right)^{d-2} \right) e^{-r(t)^2/(2t)} \end{aligned}$$

Hence, if $r(t) \geq kt^p$ for large t for some $p > \frac{1}{2}$, then

$$\int_0^\infty P_0(|X_t| > r(t)) dt < \infty.$$

2. General criterion for (2) and its application

Assume that $\mathcal{E}^{(0)}$ is recurrent. For $\Gamma \subset [0, \infty) \times X$, h : excessive function of Z_t ($h \cdot I_\Gamma \in L^2(dtdm)$)

$$H_\Gamma h(z) \equiv E_z(h(Z_{\sigma_\Gamma})).$$

$H_\Gamma h = \text{q.c.version of } e_{h \cdot I_\Gamma} = \lim_{\epsilon \rightarrow 0} h_\epsilon$:

$$- \int_X \frac{\partial h_\epsilon}{\partial t}(t, x) v(x) dm(x) + \mathcal{E}^t(h_\epsilon(t, \cdot), v) = \frac{1}{\epsilon} \int_X (h_\epsilon - h I_\Gamma)^-(t, x) v(x) dm(x)$$

Since $h_\epsilon \leq h$ on Γ ,

$$\int_X (h_\epsilon - h I_\Gamma)^-(t, x) (h_\epsilon - g)(t, x) dm(x) \leq 0,$$

for $g \geq 0$, $g = h$ on Γ . Hence

$$\begin{aligned}
& - \int_X \frac{\partial h_\epsilon}{\partial t}(t, x)(h_\epsilon - g)(t, x) dm(x) + \mathcal{E}^t(h_\epsilon(t, \cdot), (h_\epsilon - g)(t, \cdot)) \\
& = -\frac{1}{2} \int_X \frac{\partial (h_\epsilon - g)^2}{\partial t}(t, x) dm(x) - \int_X \frac{\partial g}{\partial t}(t, x)(h_\epsilon - g)(t, x) dm(x) \\
& \quad + \mathcal{E}^t(h_\epsilon(t, \cdot), (h_\epsilon - g)(t, \cdot)) \\
& \leq 0.
\end{aligned} \tag{6}$$

Multiply $\frac{1}{\underline{a}(t)}$ and integrate over $t \in [t_1, t_2]$:

$$\begin{aligned}
0 & \geq \frac{1}{2\underline{a}(t_1)} \|(h_\epsilon - g)(t_1, \cdot)\|_m^2 - \frac{1}{2\underline{a}(t_2)} \|(h_\epsilon - g)(t_2, \cdot)\|_m^2 \\
& \quad + \frac{1}{2} \int_{t_1}^{t_2} \left(\int_X (h_\epsilon - g)^2(t, x) dm(x) \right) \frac{d}{dt} \left(\frac{1}{\underline{a}(t)} \right) dt \\
& \quad - \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \left(\int_X \frac{\partial g}{\partial t}(t, x)(h_\epsilon - g)(t, x) dm(x) \right) dt \\
& \quad + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^t(h_\epsilon(t, \cdot), (h_\epsilon - g)(t, \cdot)) dt.
\end{aligned}$$

If $g(t, x)$ is non-increasing relative to t on $[t_1, t_2]$, by letting $\epsilon \rightarrow 0$,

$$\begin{aligned}
0 & \geq -\frac{1}{2\underline{a}(t_2)} \int_{X \setminus \Gamma_{t_2}} (H_\Gamma h)^2(t_2, x) dm - \frac{1}{\underline{a}(t_1)} \int_{X \setminus \Gamma_{t_1}} (g \cdot H_\Gamma h)(t_1, x) dm \\
& \quad - \frac{1}{2} \int_{t_1}^{t_2} \left(\int_{X \setminus \Gamma_t} g^2(t, x) dm(x) \right) \frac{d}{dt} \frac{1}{\underline{a}(t)} dt \\
& \quad + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^t(H_\Gamma h(t, \cdot), (H_\Gamma h - g)(t, \cdot)) dt
\end{aligned}$$

Lemma 4 Suppose that $g(t, x) = h(t, x)$ on Γ , $g(\cdot, x)$ and \underline{a} are non-increasing on $[t_1, t_2]$. Then

$$\begin{aligned}
& -\frac{1}{2\underline{a}(t_2)} \int_{X \setminus \Gamma_{t_2}} (H_\Gamma h)^2(t_2, x) dm(x) \\
& \quad - \frac{1}{\underline{a}(t_1)} \int_{X \setminus \Gamma_{t_1}} (g \cdot H_\Gamma h)(t_1, x) dm(x) \\
& \quad + \int_{t_1}^{t_2} \mathcal{E}^t(H_\Gamma h(t, \cdot), H_\Gamma h(t, \cdot)) dt \\
& \leq \frac{1}{2} \int_{t_1}^{t_2} \left(\int_{X \setminus \Gamma_t} g^2(t, x) dm(x) \right) \frac{d}{dt} \left(\frac{1}{\underline{a}(t)} \right) dt \\
& \quad + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^t(H_\Gamma h(t, \cdot), g(t, \cdot)) dt.
\end{aligned} \tag{7}$$

$C = \{x : |x| \leq \ell\}$ ($0 < \ell < 1$), $B_R = \{x : |x| \leq R\}$.

$\mathbf{M}^{T,C} = (X_t, P_{(s,x)}^{T,C})$: process corresponding to the Dirichlet form \mathcal{E}^t
with reflecting boundary $\partial B_{r(T)+1}$ and absorbing boundary ∂C .

$\xi_T(t)$: non-increasing function such that $\xi_T(t) = 1$ for $t \leq T - 1$
 $\xi_T(t) = 0$ for $t \geq T$.

Then $h(t, x) \equiv \xi_T(t)$ is an excessive function relative to $\mathbf{M}^{T,C}$.

$g_T(t, x)$: non-increasing function such that $g_T(t, x) = \xi_T(t)$ on Λ
 $g_T(t, x) = 0$ for $x \in C$, Apply Lemma 4 for $\Lambda_T = \{(t, x) : t \leq T, r(t) < |x| \leq r(T) + 1\}$.

$$\begin{aligned} & -\frac{1}{2\underline{a}(t_1)} \int_{X \setminus \Lambda_{t_1}} (H_\Lambda \xi_T)^2(t_1, x) dm(x) \\ & -\frac{1}{2\underline{a}(t_2)} \int_{X \setminus \Lambda_{t_2}} (H_\Lambda \xi_T)^2(t_2, x) dm(x) \\ & + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t)} \mathcal{E}^t (H_\Lambda \xi_T(t, \cdot), H_\Lambda \xi_T(t, \cdot)) dt \\ & \leq \frac{1}{2} \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{1}{\underline{a}(t)} \right) \left(\int_{X \setminus \Lambda_t} g_T^2(t, x) dm(x) \right) dt \\ & + \int_{t_1}^{t_2} \frac{1}{\underline{a}(t, x)} \mathcal{E}^t (H_\Lambda \xi_T(t, \cdot), g_T(t, \cdot)) dt. \end{aligned}$$

$$\begin{aligned} \Gamma & \equiv \lim_{T \rightarrow \infty} \Lambda_T = \{(t, x) : r(t) < |x|\} \\ u_\Gamma(t, x) & \equiv \lim_{T \rightarrow \infty} H_\Lambda \xi_T(t, x) = P_{(t,x)}^X (\sigma_\Gamma < \sigma_C) \\ g(t, x) & \equiv \lim_{T \rightarrow \infty} g_T(t, x) \end{aligned}$$

Then

$$\begin{aligned} & \lim_{S \rightarrow \infty} \frac{1}{S-s} \left(-\frac{1}{2\underline{a}(S)} \int_{X \setminus \Gamma_S} u_\Gamma(S, x)^2 dm(x) \right. \\ & \quad \left. + \int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^t (u_\Gamma(t, \cdot), u_\Gamma(t, \cdot)) dt \right) \\ & \leq \frac{1}{2} \lim_{S \rightarrow \infty} \frac{1}{S-s} \int_s^S \frac{d}{dt} \underline{a}(t) \left(\int_{X \setminus \Lambda_t} g_T^2(t, x) dm(x) \right) dt \\ & \quad + \lim_{S \rightarrow \infty} \frac{1}{S-s} \int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^t (u_\Gamma(t, \cdot), g(t, \cdot)) dt. \end{aligned}$$

Hence, if

$$\lim_{s \rightarrow \infty} \frac{1}{\underline{a}(S)(S-s)} \int_{X \setminus \Gamma_s} u_\Gamma(S, x)^2 dm(x) = 0 \quad (8)$$

$$\lim_{s \rightarrow \infty} \frac{1}{S-s} \int_s^S \frac{d}{dt} \left(\frac{1}{\underline{a}(t)} \right) \left(\int_{X \setminus \Gamma_t} g^2(t, x) dm(x) \right) dt = 0 \quad (9)$$

$$\lim_{s \rightarrow \infty} \frac{1}{S-s} \int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^{(X,t)}(g(t, \cdot), g(t, \cdot)) dt = 0 \quad (10)$$

then

$$\lim_{s \rightarrow \infty} \frac{1}{S-s} \int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^t(u_\Gamma(t, \cdot), u_\Gamma(t, \cdot)) dt = 0. \quad (11)$$

Put

$$\bar{a}(t, r) = \sum_{i,j=1}^d \int_{\partial B_1} a_{ij}(t, \theta r) \theta_i \theta_j d\sigma(\theta)$$

∂B_1 : unit of radius 1, $d\sigma$: surface measure

$$\psi_t(r) = \begin{cases} 0 & r \leq \ell \\ \frac{1}{A(t)} \int_\ell^r \bar{a}(t, r)^{-1} r^{1-d} dr & \ell \leq r \leq r(t) \\ 1 & r(t) \leq r \end{cases} \quad (12)$$

where

$$A(t) = \int_\ell^{r(t)} \bar{a}(t, r)^{-1} r^{1-d} dr.$$

Suppose

$$\bar{a}(t, r) \text{ and } A(t) \text{ are non-decreasing relative to } t. \quad (13)$$

Then $g(t, x) \equiv \psi_t(|x|)$ is non-increasing. Since

$$\int_{\Gamma_S} u_\Gamma(S, x)^2 dm(x) \leq \int_{\{x: |x| < r(S)\}} dm(x) \leq k r(S)^d,$$

(8) is satisfied if

$$\lim_{s \rightarrow \infty} \frac{1}{S \underline{a}(S)} r(S)^d = 0. \quad (14)$$

Similarly, since

$$\int_{X \setminus \Gamma_t} g^2(t, x) dx \leq k (r(t)^d - \ell^d),$$

(9) holds if

$$\lim_{t \rightarrow \infty} \frac{d}{dt} \left(\frac{1}{\underline{a}(t)} \right) \times r(t)^d = 0. \quad (15)$$

since

$$\mathcal{E}^{X,t}(g(t, \cdot), g(t, \cdot)) = \frac{1}{A(t)},$$

(10) is satisfied if

$$\lim_{t \rightarrow \infty} \underline{a}(t) \times A(t) = \infty. \quad (16)$$

For the existence of $r(t)$ satisfying (16), it is necessary that $\bar{a}(t, r)$ satisfies

$$A(\infty) = \int_{\ell}^{\infty} \bar{a}(t, r)^{-1} r^{1-d} dr = \infty, \quad (17)$$

(Ichihara's test for the recurrence of \mathcal{E}^t).

If (14), (15) and (16) holds, then

$$\lim_{S \rightarrow \infty} \frac{1}{S-s} \int_s^S \frac{1}{\underline{a}(t)} \mathcal{E}^t(u_{\Gamma}(t, \cdot), u_{\Gamma}(t, \cdot)) dt = 0.$$

Hence

$$\lim_{S \rightarrow \infty} \frac{1}{S-s} \int_s^S \int_X \sum_{i,j=1}^d a_{ij}^{(0)}(x) \frac{\partial u_{\Gamma}(t, x)}{\partial x_i} \frac{\partial u_{\Gamma}(t, x)}{\partial x_j} dx dt = 0.$$

Since $\mathcal{E}^{(0)}$ on X is recurrent, $\exists \varphi_0 \in L_+^1(m)$, $\exists K(\varphi_0)$, $\exists G \subset C$ such that

$$\int_X \left| u(x) - \frac{1}{m(G)} \langle I_G \cdot m, u \rangle \right| \varphi_0(x) dm(x) \leq K(\varphi_0) \left(\mathcal{E}^{(0)}(u, u) \right)^{1/2}$$

Let $u = u_{\Gamma}$, then $\langle I_G \cdot m, u_{\Gamma} \rangle = 0$,

$$\left(\int_X u_{\Gamma}(t, x) \varphi_0(x) dm(x) \right)^2 \leq (K(\varphi_0))^2 \mathcal{E}^{(0)}(u_{\Gamma}(t, \cdot), u_{\Gamma}(t, \cdot)).$$

Therefore

$$\begin{aligned} & \lim_{S \rightarrow \infty} \frac{1}{S-s} \int_s^S \left(\int_{\Gamma_t} u_{\Gamma}(t, x) \varphi_0(x) dm(x) \right)^2 dt \\ &= \lim_{S \rightarrow \infty} \frac{1}{S-s} \int_s^S \left(P_{(t, \varphi_0 \cdot m)}(\sigma_{\Gamma} < \sigma_C) \right)^2 dt = 0. \end{aligned}$$

If $\varphi_0 \in \mathcal{D}(\mathcal{G}^{(t)})$, for all t ,

$$\begin{aligned} & \int_X P_{(t,y)}(\sigma_\Gamma < \sigma_C) \varphi_0(y) dy - \int_X P_{(s,x)}(\sigma_\Gamma < \sigma_C) \varphi_0(x) dx \\ &= \int_s^t \int_X \mathcal{E}^\tau \left(P_{(\tau,\cdot)}(\sigma_\Gamma < \sigma_C), \varphi_0 \right) d\tau \\ &\leq \int_s^t \int_X |\mathcal{G}^s \varphi_0|(x) dx \end{aligned}$$

If $\sup_s \|\mathcal{G}^s \varphi_0\|_{L^1} < \infty$, then $P_{(t,\varphi_0 \cdot m)}(\sigma_\Gamma < \sigma_C)$ is uniformly continuous relative to t and hence

$$\lim_{t \rightarrow \infty} P_{(t,\varphi_0 \cdot m)}(\sigma_\Gamma < \sigma_C) = 0.$$

Theorem 5 Suppose that $\bar{a}(t, r)$ is non-decreasing relative to t . If there exists a function $r(t)$ satisfying (14), (15) and (16), then

$$\lim_{S \rightarrow \infty} \frac{1}{S-s} \int_s^S P_{(t,\varphi_0 \cdot m)}(\sigma_\Gamma < \sigma_C) dt = 0.$$

In particular, if $\sup_s \|\mathcal{G}^s \varphi_0\|_{L^1} < \infty$, then

$$\lim_{s \rightarrow \infty} P_{(s,\varphi_0 \cdot m)}(\sigma_\Gamma < \sigma_C) = 0.$$

If the result holds for some Γ , then it also holds for any subset of Γ .

Example 6 Suppose that $\underline{a}(t) = \underline{a}$ is independent of t . Then (15) is satisfied. Condition (14) holds if

$$r(t) = o\left(t^{1/d}\right). \quad (18)$$

Suppose that

$$\bar{a}(t, r) \leq b(t) \bar{a}_0(r)$$

for a non-decreasing function $b(t)$. Then

$$A(t) \geq \frac{1}{b(t)} \int_\ell^{r(t)} \bar{a}_0(r)^{-1} r^{1-d} dr.$$

Hence, if we can find $r(t)$ such that $r(t) = o(t^{1/d})$ and

$$\lim_{t \rightarrow \infty} \frac{1}{b(t)} \int_\ell^{r(t)} \bar{a}_0(r)^{-1} r^{1-d} dr = \infty, \quad (19)$$

then the conditions of Theorem hold. In particular, if $a_{ij}(t, x) = a_{ij}^{(0)}(x)$ with $(a_{ij}^{(0)}(x))$ satisfying (17) for

$$\bar{a}_0(r) = \sum_{i,j=1}^d \int_{\partial B_1} a_{ij}^{(0)}(\theta r) \theta_i \theta_j d\sigma(\theta),$$

then

$$A(t) = \int_{\ell}^{r(t)} \bar{a}_0(r)^{-1} r^{1-d} dr.$$

Hence $r(t) = t^\alpha$ with $0 < \alpha (< d/2)$ satisfies the condition of Theorem.

In the case of Brownian motion $a_{ij}(t, x) = \frac{1}{2} \delta_{ij}$. Assume $d = 1$ or 2 .

$$d = 1 \implies A(t) = r(t) - \ell$$

$$\implies r(t) = t^\alpha + 1 \text{ for } 0 < \alpha \text{ satisfies the condition of the theorem.}$$

$$d = 2 \implies A(t) = \log r(t) - \log \ell$$

$$\implies r(t) = t^\alpha \text{ for } 0 < \alpha \text{ satisfies the condition of the theorem.}$$